$12j$-symbols and four-dimensional quantum gravity

Mauro Carfora†§, Maurizio Martellini‡§,
Annalisa Marzuoli†§
†Dipartimento di Fisica Nucleare e Teorica dell'Università di Pavia
Via Bassi 6, I-27100 Pavia, Italy
‡Dipartimento di Fisica, Università di Roma I *La Sapienza*
Piazzale Aldo Moro 2, 00185 Roma, Italy
On leave of absence from:
Dipartimento di Fisica dell'Universitá di Milano,
Via Celoria, 16, I-20133 Milano, Italy
§Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Italy

Abstract

We propose a model which represents a four-dimensional version of Ponzano and Regge's three-dimensional euclidean quantum gravity. In particular we show that the exponential of the euclidean Einstein-Regge action for a $4d$-discretized block is given, in the semiclassical limit, by a gaussian integral of a suitable $12j$-symbol. Possible developments of this result are discussed.

In 1968 Ponzano and Regge[1] discovered a deep connection between the expansion of a Racah-Wigner $6j$-symbol for large values of its angular momenta and the partition function for $3d$-euclidean quantum gravity, discretized according to Regge's prescription[2]. In the early 80's other authors[3, 4] added some interesting results to the original idea, but it is just during the last year that new exciting results have been provided[5, 6, 7, 8].
Before addressing our main topic, and in order to fix notations which will be used in the following, we give a brief review of Ponzano and Regge's model. The asymptotic form of the Racah-Wigner $6j$-symbol reads[1]:

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{array} \right\} \stackrel{J_m \gg 1}{\longrightarrow} \frac{1}{\sqrt{12\pi V_T}} \cos(S_R[T] + \pi/4) \tag{1}$$

where $J_m = 0, 1/2, 1, 3/2, 2, \dots$ for each $m = 1, 2, \dots, 6$ and units are chosen for which $\hbar = G = 1$ (G is the Newton constant). The geometrical interpretation of the right-hand side of the former expression relies on the tetrahedral symmetry of the $6j$-symbol. The six quantities $(J_m + 1/2)$, $m = 1, 2, \dots, 6$, can be associated with the edges in the surface of a tetrahedron τ embedded in $\mathbf{R}^3$. T is the tetrahedron (more precisely, the 3-simplex) obtained by filling in the surface τ with a portion of flat 3-space (see fig. 1). V_T is the euclidean volume of T, and finally $S_R[T]$ is the euclidean Einstein-Regge action for T, namely[2]:

$$S_R[T] = \sum_{m=1}^{6} (J_m + 1/2)\theta_m \tag{2}$$

where θ_m is the angle between the outer normals of the two faces of T which share the m-th edge. (Recall that the euclidean Einstein-Regge action in dimension d is a discretized version[2] of the usual Einstein-Hilbert action $\int d^d x \sqrt{g} R$ for a smooth riemannian manifold $(\mathcal{M}^d, g)$ when one takes a simplicial decomposition M^d into euclidean d-simplices; the curvature is associateted with the collection of the $(d-2)$-simplices, or bones).
In the limit $J_m \gg 1$, we may disregard $1/2$ with respect to J_m in (2), and thus rewrite (1) (up to a constant phase factor) simply as:

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{array} \right\} \sim \text{"}PFP\text{"}\ (exp\,\{i \sum_{m=1}^{6} J_m \theta_m\}) = \text{"}PFP\text{"}\ (exp\,\{iS_R[T]\}) \tag{3}$$

where "PFP" means "take the positive frequency part of $(\dots)$", and the symbol $\sim$ stands, from now on, for the limit $J_m \gg 1$. Expression (3) represents the semiclassical limit of the partition function for $3d$-euclidean Einstein-Regge gravity involving just one elementary building block, namely the tetrahedron T. In order to consider the case of a generic $3d$-simplicial manifold M^3 of fixed topology, one has to perform the following steps: *i*) associate with each 3-simplex in M^3 a $6j$-symbol; *ii*) take the sum over the "internal edges" of the simplicial dissection of the product of these $6j$'s and *iii*) multiply by suitable additional factors. After a regularization, one gets the semiclassical partition function for M^3. This procedure is extensively illustrated elsewhere[1, 3, 9], so that we do not insist anymore on this point.

One of the main open problem connected with the procedure outlined above is the impossibility of extending it to higher-dimensional cases, and in particular to the physically significant case of $4d$-quantum gravity. This paper represents a first progress in this direction. In particular, we shall provide a $4d$ version of the asymptotic formula (3) involving in its right-hand side the euclidean Einstein-Regge action for a suitable $4d$-simplex. On the left-hand side there will be a gaussian integral of a $12j$-symbol according to the procedure which we are going to describe.

The $3nj$-symbols ($n = 2, 3, \ldots$) appear in the quantum theory of angular momentum when one performs the decomposition of tensor representations of the group $SU(2)$ into the direct sum of irreducible representations. Consider in particular the $12j$-symbol of the first kind (we adopt from now on the notation of Yutsis *et al.*[10]):

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} \tag{4}$$

where all j's, l's and k's run over $\{0, 1/2, 1, 3/2, 2, \ldots\}$.

Upon imposing the condition $j_4 = 0$, one gets what we shall call the *reduced* $12j$-symbol:

$$\begin{Bmatrix} j_1 & j_2 & j_3 & 0 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} \equiv \begin{Bmatrix} \textit{reduced} \\ 12j \end{Bmatrix} \tag{5}$$

where the shorthand notation in the right-hand side will be used whenever the specification of the arguments will be clear from the context.

The symmetry properties of the reduced $12j$-symbol tell us that:

$$\begin{Bmatrix} \textit{reduced} \\ 12j \end{Bmatrix} = 0 \text{ unless} : l_4 = k_1 \text{ and } l_3 = j_3 \tag{6}$$

from which we see that only 9 among the original parameters are independent.

The reduced $12j$-symbol can be related to a product of two $6j$-symbols by means of an exact relation[10] which, after a suitable change of variables, reads:

$$\begin{Bmatrix} J_1 & J_2 & J_2' & 0 \\ J_3 & J_3' & J_2' & J_5 \\ J_5 & J_4 & J_4' & J_6 \end{Bmatrix} = N\, \delta(J_4, J_5')\delta(J_2, J_1')\delta(J_6, J_6') \begin{Bmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{Bmatrix} \begin{Bmatrix} J_1' & J_2' & J_3' \\ J_4' & J_5' & J_6' \end{Bmatrix} \tag{7}$$

where:

$$N = (-1)^{J_5 - J_4 + J_3' + J_2' - J_2 + J_1} / \sqrt{J_5 J_2'} \tag{8}$$

Owing to the presence of the three Kronecker delta's in (7), among the twelve angular momenta $\{J_m, J'_n\}, (n, m = 1, 2, \ldots, 6)$, only the correct number of independent J's and J''s survives, namely:

$$\{J_1, J_2, J_3, J_4, J_5, J_6, J'_2, J'_3, J'_4\} \tag{9}$$

The form of the decomposition (7) allows to evaluate the semiclassical limit of the reduced $12j$-symbol simply by applying the result of Ponzano and Regge (3) to each $6j$-symbol. Then we have:

$$\left\{ \begin{array}{c} reduced \\ 12j \end{array} \right\} \sim \frac{N\,\delta(a;b;c)}{\sqrt{12\pi V_T} \cdot \sqrt{12\pi V_{T'}}}\; exp \left\{ i \sum_{m=1}^{6} J_m \theta_m + i \sum_{n=1}^{6} J'_n \theta'_n \right\} \tag{10}$$

where: $\sim$ stands for $J_m,\ J'_n \gg 1$, $(m, n = 1, 2, \ldots, 6)$; $a \equiv J_4 - J'_5$, $b \equiv J_2 - J'_1$, $c \equiv J_6 - J'_6$; $\delta(a;b;c) = 1$ if and only if $\delta(a) = \delta(b) = \delta(c) = 1$, and $\delta(a;b;c) = 0$ otherwise. $V_T[V_{T'}]$ is the volume of the tetrahedron $T[T']$ associated with the $6j$-symbol containing $\{J_m\}[\{J'_n\}]$; $\theta_m[\theta'_n]$ is the angle between the outer normals of the two faces of $T[T']$ sharing the edge $J_m[J'_n]$.
Notice that in (10) we have omitted "PFP of the exponential" which appeared in (3); as we shall see, this is completely consistent within the framework of our procedure. Moreover, the argument of the exponential can be rearranged in order to take into account (9). To this end, we first introduce a relabelling of the nine angular momenta, namely:

$$(J_1, J_2, J_3, J_4, J_5, J_6, J'_2, J'_3, J'_4) \rightarrow (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots, \mathcal{J}_9) \tag{11}$$

Then, according with this new enumeration, we can define:

$$(\theta_1, (\theta_2 + \theta'_1), \theta_3, (\theta_4 + \theta'_5), \theta_5, (\theta_6 + \theta'_6), \theta'_2, \theta'_3, \theta'_4) \equiv (\psi_1, \psi_2, \psi_3, \ldots, \psi_9) \tag{12}$$

With these substitutions (10) becomes:

$$\left\{ \begin{array}{c} reduced \\ 12j \end{array} \right\} \sim \tilde{N}\; exp \left\{ i \sum_{k=1}^{9} \mathcal{J}_k \psi_k \right\} \tag{13}$$

where we took into account the fact that $\delta(a;b;c) = 1$ and we set $\tilde{N} \equiv N/\sqrt{12\pi V_T} \cdot \sqrt{12\pi V_{T'}}$.
For a given real, non singular and symmetric 9×9 matrix Δ consider now the following multiple gaussian integral over the set of real variables $\{\psi_i\}$, $(i = 1, 2, \ldots, 9)$:

$$Z\,[\mathcal{J}_i\,; \Delta_{ki}] = \int \left\{ \begin{array}{c} reduced \\ 12j \end{array} \right\} exp \left\{ -\frac{1}{2} \sum_{k,i=1}^{9} \psi_k \Delta_{ki}\, \psi_i \right\} \prod_{j=1}^{9} d\psi_j \tag{14}$$

Replacing (13) in the above expression, and using the standard formula for gaussian integration, we find that in the semiclassical limit (i.e. in the limit $\mathcal{J}_i \gg 1$)the following result holds true:

$$Z\left[\mathcal{J}_i\,;\Delta_{ki}\right] \sim \hat{N}\,(det\Delta)^{-1/2}\,exp\left\{-\sum_{k,i=1}^{9}\mathcal{J}_k(\Delta^{-1})_{ki}\,\mathcal{J}_i\right\} \tag{15}$$

where $det\Delta$ and Δ^{-1} are respectively the determinant and the inverse of the matrix Δ, and we put $\hat{N} \equiv (2\pi)^6\tilde{N}$.
In order to interpret (15), recall that the form of the euclidean Einstein-Regge action for a generic $4d$-simplicial manifold M^4 with boundary ∂M^4 is (up to an arbitrary term depending on the edge lenghts in the boundary)[11]:

$$8\pi S_R[M^4] = \sum_{b\in intM^4} A(b)\epsilon(b) + \sum_{b\in\partial M^4} A(b)\alpha(b) \tag{16}$$

where b stands for "bone" (in dimension 4 a bone is a 2-simplex where, according to Regge Calculus, curvature is concentrated) and $A(b)$ is the area of b. The first sum in (16) is over the collection of bones belonging to the interior of M^4, $intM^4$, and $\epsilon(b)$ represent the so called defect angle[2] associated with the bone b. The second sum is over the bones lying in the boundary of M^4, ∂M^4, and $\alpha(b)$ has to be interpreted as the angle between the outer normals of the two boundary 3-simplices which intersect at b.
Consider again our expression (15), and in particular the sum appearing as the argument of the exponential. The terms of this sum are quadratic in the angular momenta $\mathcal{J}_i$, $(i = 1, 2, \ldots, 9)$, and thus are related to the area of some 2-dimensional geometric object. This circumstance relies on the geometrical interpretation of the reduced $12j$-symbol. Begin then by recalling that, just like in the case of a $6j$-symbol, the structure of the reduced $12j$-symbol (5) can be associated with a diagram in $\mathbf{R}^3$ (indeed, this is true for any $3nj$-symbol[1, 10]). As is easily seen from fig.2a, this diagram is given by the pair of tetrahedra T and T' joined along one of their face (*cfr.* also decomposition (7)).
The effect on the reduced $12j$-symbol of the gaussian integration performed in (14), at the semiclassical level (15), is to give rise to terms proportional to the product $J_m \cdot J'_n$, for suitable m and n (*cfr.* (11)). The crucial observation is that the existence of such contributions can be explained in a coherent way only by allowing the appearance of a tenth edge besides the nine original $\mathcal{J}_i$. The role of this new edge, the lenght of which will be denoted by L, becomes more transparent if we refer to fig.2b, where we show a representation of an euclidean 4-simplex σ which is built up from the $3d$-simplicial manifold of fig.2a by joining in $\mathbf{R}^4$ the upper and the lower vertices with an edge of lenght L. Then we see that σ has three faces, namely $(J_1LJ'_2)$, $(J_3LJ'_3)$, $(J_5LJ'_4)$, which contain the edge L and which appear in (15) in three terms, proportional to $J_1 \cdot J'_2$, $J_3 \cdot J'_3$ and $J_5 \cdot J'_4$ respectively.
Then the sum in the exponential (15) can be related to the Einstein-Regge action (16) written for the 4-simplex σ provided that we identify (up to some normalization factors,

see below) the matrix elements $(\Delta^{-1})_{ki}$ with a suitable set of angles of the type $\epsilon(b)$ and $\alpha(b)$. More precisely, we set:

$$\sum_{k,i=1}^{9} \mathcal{J}_k (\Delta^{-1})_{ki} \, \mathcal{J}_i \equiv \sum_{k,i=1}^{9} \Theta_{ki} A_{ki} \tag{17}$$

where A_{ki} is the area of the 2-face of σ containing $\mathcal{J}_k$ and $\mathcal{J}_i$ (up to combinatorial factors). According to (16), the angle Θ_{ki} has to be interpreted either as a defect angle (if the 2-face containing $\mathcal{J}_k$ and $\mathcal{J}_i$ belongs to the interior of σ), or as an angle between the outer normals of the two 3-simplices which share the $(\mathcal{J}_i, \mathcal{J}_k)$-face (if this 2-face belongs to the boundary of σ). For what concerns the explicit relation between the matrices Δ^{-1} and Θ, notice that the area A_{ki} is, for each $k, i = 1, 2, \ldots, 9$, a known function of the edges[12] and also that obviously $A_{ii} = 0$ (the reader may find elsewhere[13] more information about the properties of the matrix Θ).
Then, according to (15) and (17) we can write :

$$Z[\mathcal{J}_i\, ; \Delta_{ki}] \;\sim\; \tilde{N}(det\Delta)^{-1/2}\; exp\,\{-S_R[\sigma]\} \tag{18}$$

where $S_R[\sigma]$ is the Einstein-Regge action for the 4-simplex σ.
As we anticipated before, the right-hand side of this expression represents the semiclassical limit of the partition function for $4d$-gravity and can be considered as a direct generalization of Ponzano and Regge's result (3). There is however an important difference. In our expression the euclidean action $S_R[\sigma]$ appears in the exponential multiplyied by the exact factor needed in order to interpret the right-hand side of (18) as the semiclassical limit of an euclidean partition function. That is quite remarkable, as either in Ponzano and Regge's or in Turaev and Viro's models the presence of the imaginary unit i in front of the euclidean action has not been yet explained in a completely satisfactory way.

Let us turn now to possible developments connected to our result (18). An important further step would be to show that a scaling relation similar to the existing relation for Ponzano and Regge's model (based on the Biedenharn-Elliot identity) holds true[1]. Moreover, our result represents just the "elementary building block" for a $12j$-model of $4d$-euclidean gravity, so that another point to deal with should concern the possibility of writing a partition function for a generic $4d$-combinatorial manifold dissected into 4-simplices.
Obviously, the most important issue to address concerns the set up of a regularization procedure yielding for a consistent continuum limit of our model. In the original $3d$-case of Ponzano and Regge such step has been implemented by exploiting the Turaev and Viro's quantum $6j$-model[5, 6, 7, 8]. In this way it has been shown that the semiclassical continuum limit of the partition function of such q-model defines a naturally regularized path integral for $3d$-euclidean gravity. It is not clear if there is a suitable generalization

of these models to our case, mainly because very little (if not nothing) appears to be currently available on quantum $12j$-symbols.
We are addressing this issue, but we have not yet been able to provide a definite answer to such questions. Nonetheless we are confident that this line of attack deeply probes into the structure of 4-d euclidean quantum gravity.

Acknowledgement
We would like to thank Tullio Regge for useful conversations.

References

1 G.Ponzano and T.Regge, *Semiclassical limit of Racah coefficients*, in: Spectroscopic and Group Theoretical Methods in Physics, ed. F. Block *et al.* (North Holland, Amsterdam, 1968) pp. 1-58.

2 T.Regge, Nuovo Cimento **19** (1961) 558-571.

3 B. Hasslacher and M.J. Perry, Phys. Lett. **B 103** (1981) 21-24.

4 S.M. Lewis, Phys. Lett. **B 122** (1983) 265-267.

5 V.G. Turaev and O.Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, LOMI Preprint (1990).

6 H. Ooguri and N. Sasakura, Mod. Phys. Lett. **A 6** (1991) 3591-3600.

7 F. Archer and R.M. Williams, Phys. Lett. **B 273** (1991) 438-444.

8 S. Mizoguchi and T.Tada, Phys. Rev. Lett. **68** (1992) 1795-1798.

9 H. Ooguri, Nucl. Phys. **B 382** (1992) 276-303.

10 A.P. Yutsis, I.B. Levinson and V.V. Vanagas, *Mathematical apparatus of the theory of angular momentum* (Israel program for scientific translations, Jerusalem, 1962).

11 J.B. Hartle and R. Sorkin, Gen. Rel. Grav. **13** (1981) 541-549.

12 H.W. Hamber, in Proc. of the Les Houches Summer School 1984, ed. K. Osterwalder and R. Stora (North Holland, Amsterdam, 1986).

13 M. Roček and R.M. Williams, Phys. Lett. **104 B** (1981) 31-37.

Figure captions

fig. 1

The diagram of the $6j$-symbol is the 3-simplex T with boundary $\partial T \equiv \tau$ embedded in $\mathbf{R}^3$. (T, τ) is homeomorphic to (D^3, S^2), where D^3 is the euclidean 3-disk and S^2 is the 2-sphere.

fig. 2a

The diagram of the reduced $12j$-symbol is the $3d$-combinatorial manifold obtained by joining the two tetrahedra T and T'. Notice that the three edges of T (J_4, J_2, J_6) form a face which is glued to the face (J_5', J_1', J_6') of T'.

fig. 2b

The 4-simplex σ can be represented in $\mathbf{R}^3$ as the combinatorial manifold of fig.2a with an additional edge of lenght L connecting two vertices as indicated (the heavy line L should be thought as lying in the fourth dimension). This drawing gives however the correct vertex-edge-face scheme of σ (σ has 5 vertices, 10 edges and 10 triangular faces). σ is topologically the 4-disk in $\mathbf{R}^4$ with boundary the 3-sphere.